

DISCRETE MEAN SQUARE OF THE RIEMANN ZETA-FUNCTION OVER IMAGINARY PARTS OF ITS ZEROS

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ABSTRACT. Assume the Riemann hypothesis. In the right-hand side of the critical strip we obtain an asymptotic formula for the discrete mean square of the Riemann zeta-function over imaginary parts of its zeros.

1. INTRODUCTION

Let $s = \sigma + it$ be a complex variable. In this paper T always tends to plus infinity.

Let $N(T)$ denote the number of zeros of the Riemann zeta-function $\zeta(s)$ in the region $0 \leq \sigma \leq 1$, $0 < t \leq T$. The Riemann-von Mangoldt formula states (Titchmarsh [18, Theorem 9.4]) that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Let $\rho = \beta + i\gamma$ denote a non real zero $\zeta(s)$. The Riemann hypothesis (RH) is a conjecture that $\beta = 1/2$ for all non real zeros of the Riemann zeta-function. We prove the following two theorems.

Theorem 1. *Assume RH. Let $\sigma > 1/2$. Let $A > 0$ be as large as we like. Then there is $\delta = \delta(A) > 0$ such that, for $|t| \leq T^A$,*

$$\sum_{0 < \gamma \leq T} |\zeta(s + i\gamma)|^2 = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi} + O_{\sigma,A}(T^{1-\delta}).$$

Theorem 2. *Assume RH. Let $\sigma > 1/2$. Then, for any $\varepsilon > 0$,*

$$\sum_{0 < \gamma \leq T} |\zeta(s + i\gamma)|^2 \ll_{\sigma,\varepsilon} T \log T + |t|^\varepsilon,$$

uniformly in t .

Theorem 2 is used in Garunkštis and Laurinčikas [10] to study the discrete universality of the Riemann zeta-function over the imaginary parts of its zeros. Roughly speaking, this means that a wide class of analytic functions can be

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approximated by shifts $\zeta(s + i\gamma)$. We note that the discrete universality was proposed by Reich [15] and developed by Bagchi [1], Sander and Steuding [16]. The mentioned authors investigated the approximation of analytic functions by shifts $\zeta(s + i\tau)$, where τ takes values from arithmetic progression $\{kh : k = 0, 1, 2, \dots\}$ with a fixed $h > 0$. Dubickas and Laurinćikas [2] in place of arithmetic progressions considered the set $\{k^\alpha h : k = 0, 1, 2, \dots\}$ with a fixed $0 < \alpha < 1$.

The related discrete mean square was considered by Gonek [11]. For real α , $|\alpha| \leq (1/4\pi) \log(T/2\pi)$, he proved

$$\sum_{0 < \gamma \leq T} \left| \zeta \left(\frac{1}{2} + i \left(\gamma + \frac{2\pi\alpha}{\log(T/2\pi)} \right) \right) \right|^2 = \left(1 - \left(\frac{\sin(\pi\alpha)}{\pi\alpha} \right)^2 \right) \frac{T}{2\pi} \log^2 T + O(T \log T).$$

The error term in the last formula was improved by Fujii [7].

The similar sums away from the critical line were investigated by Laaksonen and Petridis [14]. Let $L(s, \chi_1)$ and $L(s, \chi_2)$ be the Dirichlet L -functions attached to the primitive Dirichlet characters χ_1 and χ_2 . For some fixed prime P , let

$$B(s, P) = \prod_{p \leq P} (1 - \chi_1(p)p^{-s})(1 - \chi_2(p)p^{-s}),$$

where the product runs over prime numbers p . Under RH, for fixed $1/2 < \sigma < 1$, they proved that

$$\sum_{0 < \gamma \leq T} B(\sigma + i\gamma, P) L(\sigma + i\gamma, \chi_1) \overline{L(\sigma + i\gamma, \chi_2)} \sim N(T) \sum_{n=1}^{\infty} \frac{d_n \overline{\chi_2}(n)}{n^{2\sigma}},$$

and

$$\sum_{0 < \gamma \leq T} B(\sigma + i\gamma, P) \overline{L(\sigma + i\gamma, \chi_1)} L(\sigma + i\gamma, \chi_2) \sim N(T) \sum_{n=1}^{\infty} \frac{e_n \overline{\chi_1}(n)}{n^{2\sigma}},$$

where

$$B(s, P) L(s, \chi_1) = \sum_{n=1}^{\infty} \frac{d_n}{n^s}, \quad B(s, P) L(s, \chi_2) = \sum_{n=1}^{\infty} \frac{e_n}{n^s}.$$

From this Laaksonen and Petridis [14] derived that, under RH, for a positive proportion of the non-trivial zeros of $\zeta(s)$ with $\gamma > 0$, the values of the Dirichlet L -functions $L(\sigma + i\gamma, \chi_1)$ and $L(\sigma + i\gamma, \chi_2)$ are linearly independent over \mathbb{R} .

The discrete mean value of the Dirichlet L -function at nontrivial zeros of another Dirichlet L -function were investigated by Garunkštis and Kalpokas [9]. See also Fujii [6], Steuding [17], and Garunkštis, Kalpokas, and Steuding [8].

In the next section we prove Theorems 1 and 2.

2. PROOFS

For the proof of Theorems 1 and 2 we will use the approximation of $\zeta(s)$ by a finite sum and the uniform version of Landau's formula (see Lemmas 3 and 4 below).

Lemma 3. *Assume RH. Let $\sigma > 1/2$ and $t > 0$. Then*

$$\zeta(s) = \sum_{n \leq t^\delta} \frac{1}{n^s} + O(t^{-\lambda})$$

where δ is any given positive number less than 1, and $\lambda = \lambda(\delta, \sigma) > 0$.

Proof. The lemma follows from Titchmarsh [18, Theorem 13.3]. \square

Lemma 4. *Assume RH. Let $x, T > 1$. Then*

$$\begin{aligned} \sum_{0 < \gamma \leq T} x^{\gamma i} &= -\frac{T}{2\pi} \frac{\Lambda(x)}{\sqrt{x}} + O(\sqrt{x} \log(2xT) \log \log(3x)) \\ &\quad + O\left(\frac{\log x}{\sqrt{x}} \min\left(T, \frac{x}{\langle x \rangle}\right)\right) + O\left(\frac{\log(2T)}{\sqrt{x}} \min\left(T, \frac{1}{\log x}\right)\right), \end{aligned}$$

where $\langle x \rangle$ denotes the distance from x to the nearest prime power other than x itself.

Proof. Under RH the lemma follows immediately from Gonek [12] and [13]. Note that stronger forms of Landau's formula are obtained by Fujii [4], [5] also by Ford and Zaharescu [3]. \square

To prove Theorems 1 and 2 we consider three cases: $t \geq 0$, $0 < -t \leq T$, and $0 < T < -t$. These cases correspond to Propositions 5, 6, and 7 below.

Proposition 5. *Assume RH. Let $\sigma > 1/2$, $t \geq 0$, and $0 < \delta < 1$. Then there is a positive number $\lambda = \lambda(\delta, \sigma)$ such that*

$$\begin{aligned} \sum_{\gamma \leq T} |\zeta(s + i\gamma)|^2 &= \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re\left(\frac{\zeta'}{\zeta}(s + 1/2)\right) \frac{T}{\pi} \\ &\quad + O\left(T(T+t)^{-\delta(\sigma-1/2)} + T(T+t)^{-\lambda} + (T+t)^{2\delta} + T^{1/2}(T+t)^{\delta-\lambda}\right). \end{aligned}$$

Proof. By Lemma 3, the bound

$$\sum_{0 < \gamma \leq T} (\gamma + t)^{-2\lambda} \ll (T + t)^{-2\lambda} T \log T \ll T(T + t)^{-\lambda}$$

and by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
 (1) \quad & \sum_{\gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 \\
 & =: \sum_{0 < \gamma \leq T} \sum_{n \leq (\gamma+t)^\delta} \frac{1}{n^{2\sigma}} + 2\Re \left(\sum_{0 < \gamma \leq T} \sum_{n < m \leq (\gamma+t)^\delta} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} \right) + R \\
 & =: A + R,
 \end{aligned}$$

where

$$(2) \quad R \ll A^{1/2} T^{1/2} (T+t)^{-\lambda} \log^{1/2} T + T(T+t)^{-\lambda}.$$

For the first double sum of A , we have

$$(3) \quad \sum_{0 < \gamma \leq T} \sum_{n \leq (\gamma+t)^\delta} \frac{1}{n^{2\sigma}} = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O\left(T(T+t)^{-\delta(2\sigma-1)} \log(T+t)\right).$$

The second double sum of A will require a longer consideration. Changing an order of the summation we obtain

$$(4) \quad \sum_{0 < \gamma \leq T} \sum_{n < m \leq (\gamma+t)^\delta} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} =: \sum_{n < m \leq (T+t)^\delta} \sum_{\max(0, m^{1/\delta}-t) < \gamma \leq T} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E,$$

where

$$E \ll \sum_{n < m \leq (T+t)^\delta} \sum_{\max(0, m^{1/\delta}-t) < \gamma \leq \max(0, (m+1)^{1/\delta}-t)} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma}.$$

Using Lemma 4, for $n < m \leq (T+t)^\delta$,

$$\begin{aligned}
 (5) \quad & \sum_{\max(0, m^{1/\delta}-t) < \gamma \leq T} (m/n)^{i\gamma} = -\frac{T - \max(0, m^{1/\delta}-t)}{2\pi} \frac{\Lambda(m/n)}{\sqrt{m/n}} \\
 & + O\left(\min\left(T \log T, (T+t)^{\delta/2} \log(T(T+t)) \log \log(T+t)\right)\right).
 \end{aligned}$$

Further,

$$\begin{aligned}
(6) \quad & \sum_{n < m \leq (T+t)^\delta} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} = \sum_{n < p^k n \leq (T+t)^\delta} \frac{p^{kit} \Lambda(p^k)}{(p^k n^2)^\sigma \sqrt{p^k}} \\
&= \sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \sum_{j \leq (T+t)^\delta/n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}} \\
&= - \sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \left(\frac{\zeta'}{\zeta}(\sigma - it + 1/2) + \sum_{j > (T+t)^\delta/n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}} \right) \\
&= - \sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \left(\frac{\zeta'}{\zeta}(\sigma - it + 1/2) + O\left(\left(\frac{(T+t)^\delta}{n}\right)^{1/2-\sigma}\right) \right).
\end{aligned}$$

In view of

$$\sum_{n \leq (T+t)^\delta} \frac{1}{n^{\sigma+1/2}} \ll 1 \quad \text{and} \quad \sum_{n > (T+t)^\delta} \frac{1}{n^{2\sigma}} \ll (T+t)^{\delta(1-2\sigma)},$$

we get

$$\sum_{n < m \leq (T+t)^\delta} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} = -\zeta(2\sigma) \frac{\zeta'}{\zeta}(\sigma - it + 1/2) + O((T+t)^{-\delta(\sigma-1/2)}).$$

We continue to consider the right-hand side of the formula (5). Reasoning similarly as in (6), we obtain

$$\begin{aligned}
S &=: \left| \sum_{n < m \leq (T+t)^\delta} \frac{(m/n)^{it} \max(0, m^{1/\delta} - t) \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} \right| \\
&= \left| \sum_{n < p^k n \leq (T+t)^\delta} \frac{p^{kit} \max(0, (p^k n)^{1/\delta} - t) \Lambda(p^k)}{(p^k n^2)^\sigma \sqrt{p^k}} \right| \\
&= \left| \sum_{n \leq (T+t)^\delta} \frac{1}{n^{2\sigma}} \sum_{t^\delta/n < j \leq (T+t)^\delta/n} \frac{j^{it} ((jn)^{1/\delta} - t) \Lambda(j)}{j^{\sigma+1/2}} \right| \\
&\leq \sum_{n \leq (T+t)^\delta} n^{1/\delta-2\sigma} \sum_{t^\delta/n < j \leq (T+t)^\delta/n} j^{1/\delta-\sigma-1/2} \Lambda(j).
\end{aligned}$$

In view of

$$\begin{aligned} \sum_{t^\delta/n < j \leq (T+t)^\delta/n} j^{1/\delta-\sigma-1/2} \Lambda(j) &\ll \frac{(T+t)^{1-\delta(\sigma-1/2)} - t^{1-\delta(\sigma-1/2)}}{n^{1/\delta-\sigma+1/2}} \\ &< \frac{T(T+t)^{-\delta(\sigma-1/2)}}{n^{1/\delta-\sigma+1/2}}, \end{aligned}$$

we see that $S \ll T(T+t)^{-\delta(\sigma-1/2)}$. This also gives the bound $E \ll T(T+t)^{-\delta(\sigma-1/2)}$. Finally, for the second double sum of A , we get

$$\begin{aligned} &2\Re \left(\sum_{0 < \gamma \leq T} \sum_{n < m \leq (\gamma+t)^\delta} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} \right) \\ &= \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s+1/2) \right) \frac{T}{\pi} + O(T(T+t)^{-\delta(\sigma-1/2)}) \\ &\quad + O((T+t)^{2\delta(1-\sigma)} \min(T \log T, (T+t)^{\delta/2} \log(T(T+t)) \log \log(T+t))). \end{aligned}$$

By the last formula together with formulas (1) and (3) we have

$$\begin{aligned} (7) \quad A &= \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s+1/2) \right) \frac{T}{\pi} \\ &\quad + O(T(T+t)^{-\delta(\sigma-1/2)} + (T+t)^{2\delta}). \end{aligned}$$

Then by (2) we see that $R \ll T(T+t)^{-\lambda/2} + T^{1/2}(T+t)^{\delta-\lambda/2}$. From this and formulas (1), (7), replacing $\lambda/2$ with λ , we obtain Proposition 5. \square

Proposition 6. *Assume RH. Let $\sigma > 1/2$, $0 < -t \leq T$, and $0 < \delta < 1$. Then there is a positive number $\lambda = \lambda(\delta, \sigma)$ such that*

$$\begin{aligned} \sum_{\gamma \leq T} |\zeta(s+i\gamma)|^2 &= \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s+1/2) \right) \frac{T}{\pi} \\ &\quad + O(T(T+T^\delta+t)^{-\delta(\sigma-1/2)} + T(T+T^\delta+t)^{-\lambda} + T^{2\delta} + T^{1/2+\delta}). \end{aligned}$$

Proof. The difference from the proof of Proposition 5 is that now the number $\gamma+t$ can be negative and it can be small in an absolute value. Let $\varepsilon > 0$. If $|\gamma+t|$ is small then, using the bound $\zeta(s) \ll t^{\varepsilon/3}$ (Titchmarsh [18, formula (14.2.5)]) and the Riemann-von Mangoldt formula, we have

$$\sum_{\substack{0 < \gamma \leq T \\ |t+\gamma| \leq T^\varepsilon}} |\zeta(s+i\gamma)|^2 \ll T^{2\varepsilon}.$$

Moreover, we separate the terms with negative $\gamma + t$ into a different sum. Then

$$(8) \quad \sum_{0 < \gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 = \sum_{0 < \gamma \leq -t-1} |\zeta(\sigma + i\gamma + it)|^2 \\ + \sum_{T^\varepsilon - t < \gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 + O(T^{2\varepsilon}).$$

Here and later we assume that empty sum is equal to zero.

We consider the first sum in the right-hand side of the last equality. Let

$$(9) \quad E := \left(\sum_{\gamma \leq -t-1} \sum_{n < m \leq (-t-\gamma)^\delta} - \sum_{n < m \leq (-t-1-\gamma_1)^\delta} \sum_{\gamma \leq -t-n^{1/\delta}} \right) \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} \\ \ll \sum_{n < m \leq (-t-1-\gamma_1)^\delta} \sum_{-t-(n+1)^{1/\delta} < \gamma \leq -t-n^{1/\delta}} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma}.$$

We have

$$(10) \quad \sum_{0 < \gamma \leq -t-1} |\zeta(\sigma + i\gamma + it)|^2 =: \sum_{\gamma \leq -t-1} \sum_{n \leq (-t-\gamma)^\delta} \frac{1}{n^{2\sigma}} \\ + 2\Re \left(\sum_{n < m \leq (-t-1-\gamma_1)^\delta} \sum_{\gamma \leq -t-n^{1/\delta}} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E \right) + R =: A + R,$$

where

$$R \ll A^{1/2}(1-t)^{1/2-\lambda} \log^{1/2}(1-t) + (1-t)^{1-\lambda}.$$

By the Riemann-von Mangoldt formula and the partial summation

$$(11) \quad A = \zeta(2\sigma) \frac{t}{2\pi} \log \frac{t}{2\pi e} + 2\Re \left(\sum_{n < m \leq (-t-\gamma_1)^\delta} \sum_{\gamma \leq -t-n^{1/\delta}} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E \right) \\ + O((1-t)^{1-\delta(2\sigma-1)} \log(2-t)).$$

Using Lemma 4, for $n < m \leq (-t)^\delta$, we find

$$\sum_{\gamma \leq -t-n^{1/\delta}} (m/n)^{i\gamma} = -\frac{-t-n^{1/\delta}}{2\pi} \frac{\Lambda(m/n)}{\sqrt{m/n}} + O((1-t)^{\delta/2} \log(t^2+2) \log \log(3-t)).$$

Reasoning similarly as in formula (6), we get

$$\begin{aligned} \sum_{n < m \leq (-t-\gamma_1)^\delta} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} &= \sum_{n \leq (-t-\gamma_1)^\delta} \frac{1}{n^{2\sigma}} \sum_{j \leq (-t-\gamma_1)^\delta/n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}} \\ &= -\zeta(2\sigma) \frac{\zeta'}{\zeta} (\sigma - it + 1/2) + O((1-t)^{-\delta(\sigma-1/2)}). \end{aligned}$$

Further,

$$\begin{aligned} \sum_{n < m \leq (-t-\gamma_1)^\delta} \frac{n^{1/\delta} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} &= \sum_{n \leq (-t-\gamma_1)^\delta} n^{1/\delta-2\sigma} \sum_{j \leq (-t-\gamma_1)^\delta/n} \frac{\Lambda(j)}{j^{\sigma+1/2}} \\ &\ll (1-t)^{1-\delta(2\sigma-1)}. \end{aligned}$$

Thus, $E \ll (1-t)^{1-\delta(2\sigma-1)}$ and

$$\begin{aligned} (12) \quad 2\Re \left(\sum_{n < m \leq (-t-1-\gamma_1)^\delta} \sum_{\gamma \leq -t-n^{1/\delta}} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E \right) \\ = -\zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta} (s+1/2) \right) \frac{|t|}{\pi} + O((1-t)^{1-\delta(\sigma-1/2)} + (1-t)^{2\delta}). \end{aligned}$$

For the first sum in the right-hand side of the formula (8), by formulas (10), (11), and (12), we get

$$\begin{aligned} (13) \quad \sum_{0 < \gamma \leq -t-1} |\zeta(\sigma + i\gamma + it)|^2 &= \zeta(2\sigma) \frac{|t|}{2\pi} \log \frac{|t|}{2\pi e} + \zeta(2\sigma) \frac{\zeta'}{\zeta} (s+1/2) \frac{|t|}{\pi} \\ &+ O((1-t)^{1-\delta(\sigma-1/2)} + (1-t)^{1-\lambda/2} + (1-t)^{2\delta} + (1-t)^{1/2+\delta}). \end{aligned}$$

We turn to the next sum in the right hand side of the formula (8).

$$\begin{aligned} (14) \quad \sum_{T^\varepsilon - t < \gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 &=: \sum_{T^\varepsilon - t < \gamma \leq T} \sum_{n \leq (t+\gamma)^\delta} \frac{1}{n^{2\sigma}} \\ &+ 2\Re \left(\sum_{n < m \leq (T+t)^\delta} \sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq T} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E_1 \right) + R_1 \\ &=: A_1 + R_1, \end{aligned}$$

where E_1 is an error after changing the order of the summation (analogous to E in formulas (4) and (9))

$$E_1 \ll \sum_{n < m \leq (T+t)^\delta} \sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq \max(T^\varepsilon, (m+1)^{1/\delta}) - t} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma}$$

and

$$R_1 \ll A_1^{1/2} T^{1/2} (T + T^\varepsilon + t)^{-\lambda} \log^{1/2} T + T(T + T^\varepsilon + t)^{-\lambda}.$$

Then

$$(15) \quad \begin{aligned} A_1 &= \zeta(2\sigma) \left(\frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{|t|}{2\pi} \log \frac{|t|}{2\pi e} \right) \\ &+ 2\Re \left(\sum_{n < m \leq (T+t)^\delta} \sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq T} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E_1 \right) \\ &+ O(T(T + T^\varepsilon + t)^{-\delta(2\sigma-1)} \log T). \end{aligned}$$

Lemma 4 gives, for $n < m \leq (T+t)^\delta$,

$$\sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq T} (m/n)^{i\gamma} = -\frac{T - (\max(T^\varepsilon, m^{1/\delta}) - t)}{2\pi} \frac{\Lambda(m/n)}{\sqrt{m/n}} + O(T^\delta).$$

Therefore, $E_1 \ll T(T + T^\varepsilon + t)^{-\delta(\sigma-1/2)}$ and

$$(16) \quad \begin{aligned} 2\Re \left(\sum_{n < m \leq (T+t)^\delta} \sum_{\max(T^\varepsilon, m^{1/\delta}) - t < \gamma \leq T} \frac{(m/n)^{i\gamma+it}}{(mn)^\sigma} + E_1 \right) \\ = \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s+1/2) \right) \frac{T+t}{\pi} + O(T(T + T^\varepsilon + t)^{-\delta(\sigma-1/2)} + T^{2\delta} + T^\varepsilon). \end{aligned}$$

For the second sum in the right-hand side of the formula (8), by formulas (14), (15), and (16), we get

$$(17) \quad \begin{aligned} \sum_{T^\varepsilon - t < \gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 &= \zeta(2\sigma) \left(\frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{|t|}{2\pi} \log \frac{|t|}{2\pi e} \right) \\ &+ \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s+1/2) \right) \frac{T+t}{\pi} + O(T(T + T^\varepsilon + t)^{-\delta(\sigma-1/2)}) \\ &+ O(T(T + T^\varepsilon + t)^{-\lambda/2} + T^{2\delta} + T^{1/2+\delta} + T^\varepsilon) \end{aligned}$$

We choose $\varepsilon = \delta$. Then Proposition 6 follows by formulas (8), (13), and (17). \square

Proposition 7. *Assume RH. Let $\sigma > 1/2$ and let $0 < T < -t$. Then*

$$\begin{aligned} \sum_{0 < \gamma \leq T} |\zeta(s + i\gamma)|^2 &= \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s+1/2) \right) \frac{T}{\pi} \\ &+ O(T(T^\varepsilon - T - t)^{-\delta(\sigma-1/2)} + T(T^\varepsilon - T - t)^{-\lambda} + |t|^{2\delta}) \\ &+ O(T^{1/2} |t|^\delta (T^\varepsilon - T - t)^{-\lambda}). \end{aligned}$$

Proof. We follow the proofs of Propositions 5 and 6. Let $0 < \varepsilon < 1$. We have

$$\sum_{\gamma \leq T} |\zeta(\sigma + i\gamma + it)|^2 = \sum_{\gamma \leq T - T^\varepsilon} |\zeta(\sigma + i\gamma + it)|^2 + O(|t|^{2\epsilon}).$$

Next

$$(18) \quad \sum_{\gamma \leq T - T^\varepsilon} |\zeta(\sigma + i\gamma + it)|^2 =: \sum_{0 < \gamma \leq T - T^\varepsilon} \sum_{n \leq (-t - \gamma)^\delta} \frac{1}{n^{2\sigma}} \\ + 2\Re \left(\sum_{n < m \leq (-t - \gamma_1)^\delta} \sum_{0 < \gamma \leq \min(T - T^\varepsilon, -t - n^{1/\delta})} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma} + E \right) + R =: A + R,$$

where E is an error after changing the order of the summation analogous to E in formulas (4) and (9)) and

$$R \ll A^{1/2} T^{1/2} (T^\varepsilon - T - t)^{-\lambda} \log^{1/2} T + T (T^\varepsilon - T - t)^{-\lambda}.$$

Then

$$(19) \quad A = \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + 2\Re \sum_{n < m \leq (-t - \gamma_1)^\delta} \sum_{0 < \gamma \leq \min(T - T^\varepsilon, -t - n^{1/\delta})} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma} \\ + O\left(T (T^\varepsilon - T - t)^{-\delta(2\sigma-1)} \log(|T + t| + 2)\right).$$

By Lemma 4, for $n < m \leq (-t)^\delta$, we obtain

$$\sum_{\gamma \leq \min(T - T^\varepsilon, -t - n^{1/\delta})} (m/n)^{i\gamma} = -\frac{\min(T - T^\varepsilon, -t - n^{1/\delta})}{2\pi} \frac{\Lambda(m/n)}{\sqrt{m/n}} \\ + O\left(\min(T \log T, |t|^{\delta/2} \log(|t|T) \log \log(|t|))\right).$$

In view of the last formula, we split the double sum from equality (19) in a following way.

$$\sum_{n < m \leq (-t - \gamma_1)^\delta} \sum_{0 < \gamma \leq \min(T - T^\varepsilon, -t - n^{1/\delta})} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma} \\ = \sum_{n \leq (T^\varepsilon - T - t)^\delta} \sum_{n < m \leq (-t - \gamma_1)^\delta} \sum_{0 < \gamma \leq T - T^\varepsilon} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma} \\ + \sum_{(T^\varepsilon - T - t)^\delta < n < (-t - \gamma_1)^\delta} \sum_{n < m \leq (-t - \gamma_1)^\delta} \sum_{0 < \gamma \leq -t - n^{1/\delta}} \frac{(m/n)^{i\gamma + it}}{(mn)^\sigma}.$$

Following a reasoning used in (6), we get

$$\begin{aligned}
& \sum_{n \leq (T^\varepsilon - T - t)^\delta} \sum_{n < m \leq (-t - \gamma_1)^\delta} \frac{(m/n)^{it} \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} = \sum_{n \leq (T^\varepsilon - T - t)^\delta} \sum_{n < p^k n \leq (-t - \gamma_1)^\delta} \frac{p^{kit} \Lambda(p^k)}{(p^k n^2)^\sigma \sqrt{p^k}} \\
&= \sum_{n \leq (T^\varepsilon - T - t)^\delta} \frac{1}{n^{2\sigma}} \sum_{j \leq (-t - \gamma_1)^\delta / n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}} \\
&= - \sum_{n \leq (T^\varepsilon - T - t)^\delta} \frac{1}{n^{2\sigma}} \left(\frac{\zeta'}{\zeta}(\sigma - it + 1/2) + \sum_{j > (-t - \gamma_1)^\delta / n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}} \right) \\
&= - \sum_{n \leq (T^\varepsilon - T - t)^\delta} \frac{1}{n^{2\sigma}} \left(\frac{\zeta'}{\zeta}(\sigma - it + 1/2) + O\left(\left(\frac{(-t - \gamma_1)^\delta}{n} \right)^{1/2-\sigma} \right) \right) \\
&= -\zeta(2\sigma) \frac{\zeta'}{\zeta}(\sigma - it + 1/2) + O((T^\varepsilon - T - t)^{-\delta(2\sigma-1)} + (-t)^{-\delta(\sigma-1/2)})
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{(T^\varepsilon - T - t)^\delta < n < (-t - \gamma_1)^\delta} \sum_{n < m \leq (-t - \gamma_1)^\delta} \frac{(m/n)^{it} (-t - n^{1/\delta}) \Lambda(m/n)}{(mn)^\sigma \sqrt{m/n}} \\
&= \sum_{(T^\varepsilon - T - t)^\delta < n < (-t - \gamma_1)^\delta} \frac{(-t - n^{1/\delta})}{n^{2\sigma}} \sum_{j \leq (-t - \gamma_1)^\delta / n} \frac{j^{it} \Lambda(j)}{j^{\sigma+1/2}} \\
&\ll T \sum_{(T^\varepsilon - T - t)^\delta < n < (-t - \gamma_1)^\delta} \frac{1}{n^{2\sigma}} \ll T (T^\varepsilon - T - t)^{-\delta(2\sigma-1)}.
\end{aligned}$$

In view of above,

$$\begin{aligned}
A &= \zeta(2\sigma) \frac{T}{2\pi} \log \frac{T}{2\pi e} + \zeta(2\sigma) \Re \left(\frac{\zeta'}{\zeta}(s + 1/2) \right) \frac{T}{\pi} \\
&\quad + O\left(T (T^\varepsilon - T - t)^{-\delta(\sigma-1/2)} + |t|^{2\delta} \right).
\end{aligned}$$

The last equality together with formula (18) proves Proposition 7. □

Proof of Theorems 1 and 2. The theorems immediately follow by Propositions 5, 6, and 7. □

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